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INVISCID HYPERSONIC FLOW
FOR POWER-LAW SHOCK WAVES

By James T. Lee

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ABSTRACT

The blast wave solutions for the axisymmetric flow fields associated with a family of power-law shocks, $r_s \sim x^n$, have been modified to properly account for the entropy layer near the surface of the body. The method of inner and outer expansions was used. It was shown that this method is of limited usefulness for n only slightly greater than 1/2 due to the slow convergence of the inner expansions, but for $(\gamma + 2)/(3\gamma + 2) \leq n < 1$ the expansions converge rapidly enough for the method to be useful. The case n = 1/2 is singular and causes no difficulty.

In the range $(\gamma + 1)/(2\gamma + 1) \leq n < 1$ the body shape given by blast wave theory is correct, but the blast wave flow field profiles must be modified near the body. For $n < (\gamma + 1)/(2\gamma + 1)$ both the body contour and the flow field profiles must be modified.

CONTENTS

SYMBOLS

A   Arbitrary constant used to define shock shape

D   Expansion term for density in outer region

F   Pressure function from blast wave theory

G   Parameter defined by Equation (2.8)

g   General symbol for dependent variables

I   Function defined by Equation (3.57)

k   Constant in defining Von Mises variable

$$k = (4n^2)^{n/1-n}$$

L   Characteristic dimension of flow field, Equation (2.13)

l Length of body

M   Mach number

n   Exponent in equation of shock

P   Expansion term for pressure in outer region

p   Pressure

u   Axial velocity

U   Expansion term for axial velocity in outer region

v   Transverse velocity

V   Expansion term for transverse velocity in outer region

x   Axial coordinate

y   Transverse coordinate

y*   Proportional distance across shock layer, Equation (4.10)

Y   Expansion term for space variable in outer region

z   Similarity variable, $z = y/y_s$

GREEK SYMBOLS

α	Parameter defined by Equation (4.4)
β	Density function from blast wave theory
γ	Ratio of specific heats
θ	Angle at which streamline crosses shock wave
ρ	Density
τ	Body thickness ratio
ϕ	Transverse velocity function from blast wave theory
Ψ	Streamfunction
ψ	Von Mises variable, $\psi = k\Psi$
ω	Independent variable in outer region, $\omega = \psi/x^{2n}$

SUBSCRIPTS

b	Value at body surface
s	Value at shock
∞	Free-stream conditions
1, 2, $\cdots$	First and second order terms in expansion

SUPERSCRIPTS

Dimensional quantity

1. INTRODUCTION

The inviscid hypersonic flow of a perfect gas about slender blunt nose bodies has been the subject of numerous investigations. Blast wave theory has proven to be a useful method for analyzing this problem. This approach uses the fact that, subject to the hypersonic, small disturbance conditions, the flow field about a blunt nose slender body is similar to the unsteady flow field of an intense blast wave propagating into a medium at rest. Lees and Kubota (Reference 1) and Kubota (Reference 2) have used the blast wave analogy to analyze the inverse problem of a family of power-law shocks, $y_s \sim x^n$. Solutions were obtained for $1/2 \leq n \leq 1$. A paraboloidal shock, $n = 1/2$, was found to give a zero thickness afterbody with all of the drag concentrated at the blunt nose. These solutions, however, do not properly account for the layer of high entropy air surrounding the body. This entropy layer consists of streamlines that pass through the blunt portion of the shock. Singularities such as infinite temperature and zero density at the body surface were obtained.

Sychev (Reference 3) introduced the idea of obtaining a corrected body shape which accounts for the presence of the entropy layer and produces the outer flow field and shock wave given by blast wave theory. He obtained a numerical solution for the body that produces a paraboloidal shock and showed that a critical value of n exists above which the body shape is not affected by the entropy layer.

Yakura (Reference 4) has applied the method of inner and outer expansions as developed by Lagerstrom, Kaplun, and Cole (References 5 and 6) for treating singular perturbation problems to the inverse problem with specified shock shapes. He obtained solutions for blunted wedges, blunted cones, and a paraboloidal shock. The paraboloidal shock gives a body which grows as $x^{1/2\gamma}$ in the downstream direction instead of the zero thickness afterbody predicted by blast wave theory.

The present analysis uses the methods developed by Yakura to add an entropy layer to the blast wave solutions obtained by Kubota for a family of power-law shocks. Only the axisymmetric case is considered, but the methods are applicable to plane flow.

2. METHOD OF ANALYSIS

2.1 Equations and Boundary Conditions

In considering the inverse problem with the shock wave specified it is convenient to use the Von Mises transformation. The independent variables are x, a downstream coordinate, and ψ, a variable that is proportional to the streamfunction. These variables make it easy to identify the body as the ψ = 0 streamline, and the conservation of entropy along streamlines helps to define the entropy layer.

Following the analysis of Yakura, Reference 4, the streamfunction for axisymmetric flow is defined by:

$$\frac{\partial \Psi}{\partial y} = \rho u y, \quad \frac{\partial \Psi}{\partial x} = -\rho v y \qquad (2.1)$$

Introducing the Von Mises variables x and ψ = kΨ, where k is an arbitrary constant that can be chosen for convenience, the hypersonic equations of motion for a perfect gas may be written as:

$$\frac{\partial y}{\partial x} = \frac{v}{u} \qquad (2.2)$$

$$y \frac{\partial y}{\partial \psi} = \frac{k}{\rho u} \qquad (2.3)$$

$$\frac{\partial v}{\partial x} + \frac{y}{k} \frac{\partial p}{\partial \psi} = 0 \qquad (2.4)$$

$$u^2 + v^2 + \frac{2\gamma}{\gamma - 1} \frac{p}{\rho} = 1 \qquad (2.5)$$

$$\frac{p}{\rho^\gamma} = 2G \sin^2 \theta \qquad (2.6)$$

The coordinate system is shown in Figure 1. In Equations (2.1) through (2.6) the variables are nondimensionalized as follows:

$$\rho = \frac{\rho'}{\rho_\infty} \qquad u, v = \frac{u', v'}{u_\infty}$$

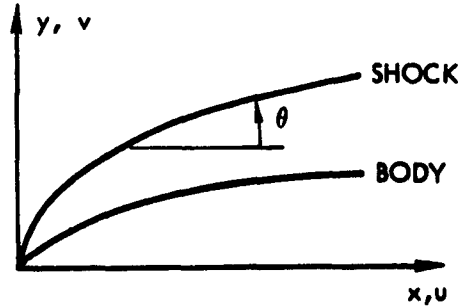$$p = \frac{p'}{\rho_\infty u_\infty^2} \qquad x, y = \frac{x', y'}{L}$$

(2.7)

Figure 1. Coordinate System

The parameter G is a function of γ,

$$G = \left(\frac{\gamma - 1}{\gamma + 1}\right)^\gamma \frac{1}{\gamma + 1}$$

(2.8)

θ is the angle at which a streamline crosses the shock, and L is a characteristic dimension of the flow field. The energy equation, Equation (2.5), has been written in integrated form. The axial momentum equation,

$$u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{\gamma v}{k} \frac{\partial p}{\partial \psi} = 0$$

(2.9)

has been replaced by Equation (2.6) which states the condition of constant entropy along streamlines. The usual hypersonic approximation

$$\left(M_\infty \theta\right)^2 >> 1 \qquad \theta << 1$$

(2.10)

is made throughout this report such that the solution is independent of free-stream Mach number.

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The boundary conditions at the shock may be written as:

$$p_s = \frac{2}{\gamma + 1} \sin^2 \theta$$

$$\rho_s = \frac{\gamma + 1}{\gamma - 1}$$

$$u_s = \left(\frac{\gamma - 1}{\gamma + 1}\right) \sin^2 \theta + \cos^2 \theta \qquad (2.11)$$

$$v_s = \frac{2}{\gamma + 1} \sin \theta \cos \theta$$

$$y_s = y_s(x)$$

where the shock shape, $y_s(x)$, is a specified function for the inverse problem.

We consider the family of power-law shocks

$$y_s' = Ax'^n \qquad (2.12)$$

where A is an arbitrary constant. The remainder of this analysis therefore reduces to that of Yakura (Reference 4) for the spacial case n = 1/2. A characteristic dimension is defined by

$$L = \left(\frac{A}{\sqrt{2}}\right)^{1/(1-n)} \qquad (2.13)$$

such that the equation for the family of shocks is

$$y_s = \sqrt{2} \, x^n \qquad (2.14)$$

For n = 1/2, L is the nose radius of the shock, but the nose radius is zero for $1/2 < n < 1$. However, the physical significance of L for $1/2 \leq n < 1$ can be seen by noting that, for $y_s = 1$

$$\frac{dy_s}{dx} \sim 0(1) \quad \text{and} \quad x \sim 0(1) \qquad (2.15)$$

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Thus L is a measure of the radius of the strong blunt portion of the shock in the nose region. The $\sqrt{2}$ term is included only for convenience in matching Yakura's result for $n = 1/2$ (Reference 4).

Continuity considerations require that:

$$\psi_s = \frac{y_s^2}{2k} = \frac{x^{2n}}{k} \tag{2.16}$$

Using this relation and taking

$$k = (4n^2)^{n/(1-n)} \tag{2.17}$$

for convenience, the condition of constant entropy along streamlines, Equation (2.6), can be expressed as:

$$\frac{p}{\rho^\gamma} = \frac{2G}{1 + 2\psi^{(1-n)/n}} \tag{2.18}$$

Equation (2.18) illustrates the convenience of using the Von Mises transformation for the inverse problem since the entropy function is known in terms of the dependent variable ψ. The boundary conditions, Equations (2.11), can be expressed as:

$$P_s = \frac{2}{\gamma + 1} \frac{1}{1 + 2\psi_s^{(1-n)/n}}$$

$$\rho_s = \frac{\gamma + 1}{\gamma - 1}$$

$$u_s = \frac{\gamma - 1}{\gamma + 1} \frac{1}{1 + 2\psi_s^{(1-n)/n}} + \frac{2\psi_s^{(1-n)/n}}{1 + 2\psi_s^{(1-n)/n}} \tag{2.19}$$

$$v_s = \frac{2}{\gamma + 1} \frac{\left(2\psi_s^{(1-n)/n}\right)^2}{1 + 2\psi_s^{(1-n)/n}}$$

$$y_s = (2k\psi_s)^{1/2}$$

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The problem is now specified by Equations (2.2), (2.3), (2.4), (2.5), (2.18) and the above boundary conditions for p, ρ, u, v, y as functions of x and ψ.

2.2 Method of Inner and Outer Expansions

The method of inner and outer expansions was developed by Lagerstrom et.al., (References 5 and 6) for treating singular perturbation problems with particular emphasis on viscous flow at high and low Reynolds number. Van Dyke (Reference 7) used the method to identify certain second order boundary layer effects such as curvature, vorticity interaction, etc. Yakura (Reference 4) has used the method to analyze the inviscid hypersonic flow about slender blunt nose bodies accounting for the entropy layer near the body. Essentially, the method divides the flow field into two regions in which different asymptotic expansions are valid and uses a limiting process to match the expansions in an overlap region.

The method as applicable to the inviscid, blunt nose, slender body problem is described below in a way that permits an understanding of the present analysis. A more general and detailed description may be found in the above references.

The outer region of the flow field is defined as consisting of those streamlines which cross the weak portion of the shock where $x \gg 1$. In this region $\psi \sim 0(\psi_s) \sim 0(x^{2n})$. An independent variable of order one is defined for the outer region by

$$\omega = \frac{\psi}{x^{2n}} \sim 0(1) \qquad (2.20)$$

and an outer expansion for the dependent variables is assumed of the form

$$0ex\ g(x, \psi) = \sum_{i=1} E_i(x) F_i(\omega) \qquad (2.21)$$

The functions $E_i(x)$ can be chosen to match the known shock conditions expanded for large x, and the boundary conditions on $F_i(\omega)$ are taken at the shock. This expansion scheme leads to an outer solution that is asymptotically valid for large distances downstream from the nose.

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Streamlines near the body cross the strong portion of the shock near the blunt nose and define an inner region where $\psi \sim 0(1)$. The expansion assumed above where $E_i(x)$ is taken from the shock conditions for large x is not valid. An expansion of the following form is assumed for the inner region:

$$\text{Iex } g(x, \psi) = \sum_{i=1} e_i(x) f_i(\psi) \qquad (2.22)$$

Since the body shape is unknown a priori, the form of $e_i(x)$ and the boundary conditions on $f_i(\psi)$ must be taken from the asymptotic behavior of the outer solution as the inner region is approached. Two limit processes are defined:

The Inner Limit of the Outer Expansion

$$\text{Ilim Oex } g(x, \psi) = \lim_{x \to \infty, \ \psi \text{ fixed}} \left[\text{Oex } g(x, \psi) \right] \qquad (2.23)$$

The Outer Limit of the Inner Expansion

$$\text{Olim Iex } g(x, \psi) = \lim_{x \to \infty, \ \omega \text{ fixed}} \left[\text{Iex } g(x, \psi) \right] \qquad (2.24)$$

If the outer solution is known, the $e_i(x)$ terms may be determined by taking the inner limit of the outer expansion. The assumed inner expansion, Equation (2.22), may then be used, and the boundary conditions on $f_i(\psi)$ are taken such that:

$$\text{Olim Iex } g(x, \psi) = \text{Ilim Oex } g(x, \psi) \qquad (2.25)$$

The outer and inner solutions can be combined in such a way as to provide a composite expansion that is uniformly valid across the entire shock layer. Essentially it is desired to eliminate the incorrect portions of both the outer and inner expansions. This can be done in the following manner:

$$g(x, \psi) = \text{Oex } g(x, \psi) + \text{Iex } g(x, \psi) - \text{Olim Iex } g(x, \psi) \qquad (2.26)$$

Note that, due to the boundary condition imposed upon the inner solution, the outer and inner limits of the composite expansion reduce to the outer and inner expansions respectively.

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3. SOLUTIONS FOR POWER-LAW SHOCKS

3.1 Outer Solution

Introducing the variable $\omega = \psi/x^{2n}$ into Equations (2.2) through (2.5), (2.18), and (2.19) and expanding for large x, the equations and boundary conditions for the outer region can be expressed as follows:

$$\frac{\partial y}{\partial x} - 2n \frac{\omega}{x} \frac{\partial y}{\partial \omega} = \frac{v}{u} \tag{3.1}$$

$$\frac{y}{x^{2n}} \frac{\partial y}{\partial \omega} = \frac{k}{\rho u} \tag{3.2}$$

$$\frac{\partial v}{\partial x} - 2n \frac{\omega}{x} \frac{\partial v}{\partial \omega} + \frac{y}{kx^{2n}} \frac{\partial p}{\partial \omega} = 0 \tag{3.3}$$

$$u^2 + v^2 + \frac{2\gamma}{\gamma - 1} \frac{p}{\rho} = 1 \tag{3.4}$$

$$\frac{p}{\rho^\gamma} = \frac{G}{\left(\omega x^{2n}\right)^{(1-n)/n}} \left[1 - \frac{1}{2\left(\omega x^{2n}\right)^{(1-n)/n}} + \cdots \right] \tag{3.5}$$

with boundary conditions at the shock, $\omega_s = 1/k$:

$$p_s = \frac{1}{(\gamma + 1)\left(\omega_s x^{2n}\right)^{(1-n)/n}} \left[1 - \frac{1}{2\left(\omega_s x^{2n}\right)^{(1-n)/n}} + \cdots \right]$$

$$\rho_s = \frac{\gamma + 1}{\gamma - 1}$$

$$u_s = 1 - \frac{1}{\gamma + 1} \frac{1}{\left(\omega_s x^{2n}\right)^{(1-n)/n}} + \frac{1}{\gamma + 1} \frac{1}{2\left(\omega_s x^{2n}\right)^{2(1-n)/n}} + \cdots \tag{3.6}$$

$$v_s = \frac{\sqrt{2}}{(\gamma + 1)\left(\omega_s x^{2n}\right)^{(1-n)/2n}} \left[1 - \frac{1}{2\left(\omega_s x^{2n}\right)^{(1-n)/n}} + \cdots \right]$$

$$y_s = (2k\omega_s)^{1/2} x^n$$

The shock conditions suggest the following form for the outer expansions:

$$p = P_1(\omega)x^{-2(1-n)} + P_2(\omega)x^{-4(1-n)} + \ldots$$

$$\rho = D_1(\omega) + D_2(\omega)x^{-2(1-n)}$$

(3.7)

$$u = 1 + U_1(\omega)x^{-2(1-n)} + U_2(\omega)x^{-4(1-n)} + \ldots$$

$$v = V_1(\omega)x^{-1(1-n)} + V_2(\omega)x^{-3(1-n)} + \ldots$$

$$y = Y_1(\omega)x^{n} + Y_2(\omega)x^{-(2-3n)} + \ldots$$

Substituting these expansions into the outer equations, Equations (3.1) through (3.5), the following set of first order outer equations and boundary conditions are obtained:

$$Y_1 - 2\omega \frac{dY_1}{d\omega} = \frac{V_1}{n}$$

(3.8)

$$Y_1 \frac{dY_1}{d\omega} = \frac{k}{D_1}$$

(3.9)

$$V_1 + \frac{2n}{(1-n)^2 k} \omega \frac{dV_1}{d\omega} Y_1 \frac{dP_1}{d\omega} = 0$$

(3.10)

$$\frac{P_1}{D_1^{\gamma}} = \frac{G}{\omega^{(1-n)/n}}$$

(3.11)

$$2U_1 + V_1^2 + \frac{2\gamma}{\gamma - 1} \frac{P_1}{D_1} = 0$$

(3.12)

with boundary conditions at the shock, $\omega_s = 1/k$:

$$P_1 = \frac{4n^2}{\gamma + 1}$$

$$D_1 = \frac{\gamma + 1}{\gamma - 1}$$

$$U_1 = -\frac{4n^2}{\gamma + 1}$$ (3.13)

$$V_1 = \frac{(2)^{3/2} n}{\gamma + 1}$$

$$Y_1 = \sqrt{2}$$

Note that the first four of the above equations do not contain the axial velocity, U_1, and can therefore be solved independently of the energy equation, Equation (3.12). This corresponds to the blast wave approximation of $u \approx u_\infty$. The axial velocity can be obtained from Equation (3.12) after the first four equations are solved. The solution of Equations (3.8) through (3.11) can be taken directly from Kubota's blast wave analysis (Reference 2) in terms of the similarity variable $z = y/y_s$:

$$P_1 = 2n^2 F(z)$$ (3.14)

$$D_1 = \beta(z)$$ (3.15)

$$V_1 = \sqrt{2} \, n\phi(z)$$ (3.16)

$$Y_1 = \sqrt{2} \, z$$ (3.17)

The functions $F(z)$, $\beta(z)$, $\phi(z)$ are given in numerical form in Reference 2 for $\gamma = 1.4$ and $n = 1/2$, 4/7, 2/3, 3/4, and 1. An approximate analytic solution for these functions is given by Mirels (Reference 8).

Finally, the first order outer solution or the blast wave solution is given by:

$$p = 2n^2 F(z) \, x^{-2(1-n)} \tag{3.18}$$

$$\rho = \beta(z) \tag{3.19}$$

$$v = \sqrt{2} \, n\phi(z) \, x^{-(1-n)} \tag{3.20}$$

$$u = 1 - n^2 \left[\frac{2\gamma}{\gamma - 1} \frac{F(z)}{\beta(z)} + \phi^2(z) \right] x^{-2(1-n)} \tag{3.21}$$

$$y = \sqrt{2} \, z x^{n} \tag{3.22}$$

The above results are not valid near the blunt nose or very far downstream where the hypersonic, small disturbance conditions are not satisfied. They are also invalid near the body and give zero density and infinite temperature on the surface of the body. Also, for $n = 1/2$, the body is of zero thickness. These last two difficulties can be eliminated by using a different expansion near the body in the inner region or entropy layer.

3.2 Inner Solution

The outer expansion assumes $\psi \gg 1$, and it can be seen from Equation (2.18) that this leads to a singularity at the body surface. Different expansions must be used for the inner region where $\psi \sim 0(1)$. The expansion of Equation (2.22) is valid if the $e_i(x)$ terms are determined by taking the inner limit of the outer expansion according to Equation (2.23). Using this expansion, a solution of Equations (2.2) through (2.5) and (2.18) can be obtained in terms of the variables x and ψ.

To obtain the Ilim 0ex $g(x, \psi)$ it is necessary to know the outer solution in analytic form near the body. This can be obtained by expanding the outer solution for the pressure in a Taylor series about the body, $\omega = 0$, and using the resulting pressure distribution to solve the remaining outer equations. Thus:

$$p = \left[P_1(0) + \left(\frac{dP_1}{d\omega} \right)_0 \omega + \cdots \right] x^{-2(1-n)} + P_2(\omega) \, x^{-4(1-n)} + \cdots \tag{3.23}$$

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Since $\psi \sim 0(1)$ in the inner region, $\omega \sim 0(1/x^{2n})$ and Equation (3.23) becomes:

$$p \approx P_1(0)x^{-2(1-n)} + \left(\frac{dP_1}{d\omega}\right)_0 x^{-2} + \cdots + P_2(\omega)x^{-4(1-n)} + \cdots \quad (3.24)$$

where $(dP_1/d\omega)_0$ is finite and of order one. From Equation (3.24) it is seen that to the accuracy involved in the first order solution the pressure may be taken as constant across the inner region and equal to the value given by the outer solution at the body surface. Substituting

$$P_1(\omega) = P_1(0) \quad (3.25)$$

into the first order outer equations gives the following results valid near the body:

$$D_1(\omega) = \left(\frac{P_1(0)}{G}\right)^{1/\gamma} \omega^{(1-n)/n\gamma} \quad (3.26)$$

$$Y_1(\omega) = \left[\frac{2k\gamma}{\gamma + 1 - \frac{1}{n}} \left(\frac{G}{P_1(0)}\right)^{1/\gamma} \omega^{\frac{1}{\gamma}\left(\gamma + 1 - \frac{1}{n}\right)} + Y_1(0)^2\right]^{1/2} \quad 3.27)$$

$$V_1(\omega) = n\left(Y_1 - 2\omega \frac{dY_1}{d\omega}\right) \quad (3.28)$$

$$U_1(\omega) = -\left(\frac{\gamma}{\gamma - 1}\frac{P_1}{D_1} + \frac{V_1^2}{2}\right) \quad (3.29)$$

Using the above relations, the inner limit process, Equation (2.23), can be applied to the outer solution. The result in terms of the inner variables can be written as:

$$\text{llim } 0\text{ex } p(x, \psi) = P_1(0) x^{-2(1-n)} \quad (3.30)$$

$$\text{Ilim Oex } \rho(x, \psi) = \left(\frac{P_1(0)}{G}\right)^{1/\gamma} \psi^{(1-n)/n\gamma} x^{-2(1-n)/\gamma} \tag{3.31}$$

$$\text{Ilim Oex } y(x, \psi) = \left\{\frac{2k\gamma}{\gamma + 1 - 1/n}\left[\frac{G}{P_1(0)}\right]^{1/\gamma} \psi^{\frac{1}{\gamma}\left(\gamma+1-\frac{1}{n}\right)} x^{\frac{2(1-n)}{\gamma}} + Y_1(0)^2 x^{2n}\right\}^{1/2} \tag{3.32}$$

Similar expressions can be obtained for v and u.

An important feature of the application of the method of inner and outer expansions to this problem can be seen from Equation (3.32). For $n = 1/2$, $Y_1(0) \equiv 0$ (i.e., the body given by the outer solution has zero thickness), and the Ilim Oex $y(x, \psi)$ reduces to:

$$\text{Ilim Oex } y(x, \psi) = \left\{\frac{2k\gamma}{\gamma - 1}\left[\frac{G}{P_1(0)}\right]^{1/\gamma}\right\}^{1/2} \psi^{\frac{\gamma-1}{2\gamma}} x^{\frac{1}{2\gamma}} \tag{3.33}$$

Equation (3.33) gives the following form for the inner expansion which is the same as that used by Yakura (Reference 2):

$$\text{Iex } y(x, \psi) = Y_1(\psi) x^{1/2\gamma} \tag{3.34}$$

For $n > 1/2$, however, $Y_1(0) \neq 0$, and Equation (3.32) must be expanded resulting in:

$$\text{Ilim Oex } y(x, \psi) = Y_1(0) x^n$$

$$+ \left\{\frac{k\gamma}{\left(\gamma + 1 - \frac{1}{n}\right)Y_1(0)}\left[\frac{G}{P_1(0)}\right]^{1/\gamma} \psi^{\frac{1}{\gamma}\left(\gamma+1-\frac{1}{n}\right)}\right\} x^{-n+\frac{2(1-n)}{\gamma}}$$

$$+ 0\left[x^{-3n+\frac{4(1-n)}{\gamma}}\right] + \cdots + Y_2(\omega)x^{-(2-3n)} + \cdots \tag{3.35}$$

where the term $Y_2(\omega) x^{-(2-3n)}$ represents the contribution of the second order outer solution. Equation (3.35) gives the following form for the inner expansion:

$$\text{Iex } y(x, \psi) = y_1(\psi) x^n + y_2(\psi) x^{-n+\frac{2(1-n)}{\gamma}} + y_3(\psi) x^{-3n+\frac{4(1-n)}{\gamma}}$$

$$+ \cdots + 0\left[x^{-(2-3n)}\right] + \cdots \tag{3.36}$$

To obtain an inner solution that is consistent with the first order outer solution, enough terms must be retained in Equation (3.36) that the order of the dropped terms is less than or equal to $x^{-(2-3n)}$. It can be seen that, as $n \to 1/2$, an increasing number of terms must be retained in the first order inner expansion. Also, for $n = 1/2$, $Y_1(0) = 0$ and Equation (3.35) is not valid. Thus Equation (3.36) can be used for all $n \geqslant 1/2$ if enough terms are retained in the expansion.

The following forms of the inner expansions are consistent with the first order outer solution in the range of values of n indicated. The $n = 1/2$ case was solved previously by Yakura (Reference 2) and is included here only for the sake of completeness.

$\dfrac{1}{2} \leqslant n < 1 \qquad p(x, \psi) = p_1(\psi) x^{-2(1-n)}$

$$\rho(x, \psi) = \rho_1(\psi) x^{\frac{-2(1-n)}{\gamma}} \tag{3.37}$$

$$u(x, \psi) = 1 + u_1(\psi) x^{-2(1-n)\frac{\gamma-1}{\gamma}}$$

$\dfrac{\gamma+1}{2\gamma+1} \leqslant n < 1 \qquad y(x, \psi) = y_1(\psi) x^n$

$$\tag{3.38}$$

$$v(x, \psi) = v_1(\psi) x^{-(1-n)}$$

$\dfrac{\gamma+2}{3\gamma+2} \leqslant n < \dfrac{\gamma+1}{2\gamma+1} \qquad y(x, \psi) = y_1(\psi) x^n + y_2(\psi) x^{-n+\frac{2(1-n)}{\gamma}}$

$$\tag{3.39}$$

$$v(x, \psi) = v_1(\psi) x^{-(1-n)} + v_2(\psi) x^{-\left(1+n+\frac{2n}{\gamma}-\frac{2}{\gamma}\right)}$$

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$$\underline{n = \frac{1}{2}} \qquad y(x, \psi) = y_1(\psi)\, x^{1/(2\gamma)}$$

$$v(x, \psi) = v_1(\psi)\, x^{-\frac{2\gamma - 1}{2\gamma}}$$

(3.40)

For $1/2 < n < (\gamma + 2)/(3\gamma + 2)$ additional terms would be required in the expansions for y and v. Only the above intervals are considered in the remainder of this report.

Substituting the above expansions into the complete hypersonic equations, Equations (2.2) through (2.5) and (2.18), results in the following system of first order inner equations:

$$\underline{\frac{\gamma + 1}{2\gamma + 1} \leq n < 1} \qquad\qquad \frac{dp_1}{d\psi} = 0 \tag{3.41}$$

$$\frac{dy_1}{d\psi} = 0 \tag{3.42}$$

$$v_1 - ny_1 = 0 \tag{3.43}$$

$$\frac{p_1}{\rho_1^{\gamma}} = \frac{2G}{1 + 2\psi^{(1-n)/n}} \tag{3.44}$$

$$u_1 = -\frac{\gamma}{\gamma - 1}\, \frac{p_1}{\rho_1} \tag{3.45}$$

$$\underline{\frac{\gamma + 2}{3\gamma + 2} \leq n < \frac{\gamma + 1}{2\gamma + 1}} \qquad \text{Two additional equations are required:}$$

$$\frac{dy_2}{d\psi} = \frac{k}{\rho_1 y_1} \tag{3.46}$$

$$v_2 - \left[\frac{2(1 - n)}{\gamma} - n \right] y_2 = 0 \tag{3.47}$$

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$\underline{n = 1/2}$ Equations (3.41), (3.44), and (3.45), plus the following two equations are required:

$$\frac{dy_1}{d\psi} = \frac{1}{\rho_1 y_1} \tag{3.48}$$

$$y_1 = 2\gamma v_1 \tag{3.49}$$

The boundary conditions are such that:

$$\text{Ilim Oex } g(x, \psi) = \text{Olim Iex } g(x, \psi) \tag{3.50}$$

The above system of equations are easily solved and the first order inner solution can be written as:

$\frac{1}{2} \le n < 1$ $p(x, \psi) = P_1(0) x^{-2(1-n)}$ $\tag{3.51}$

$$\rho(x, \psi) = \left[\frac{P_1(0)}{2G}\right]^{1/\gamma} \left(1 + 2\psi^{(1-n)/n}\right)^{1/\gamma} x^{-2(1-n)/\gamma} \tag{3.52}$$

$$u(x, \psi) = 1 - \frac{\gamma}{\gamma - 1} P_1(0) \left[\frac{2G}{P_1(0)}\right]^{1/\gamma} \frac{1}{\left(1 + 2\psi^{(1-n)/n}\right)^{1/\gamma}} x^{-2(1-n)\frac{\gamma-1}{\gamma}} \tag{3.53}$$

$\frac{\gamma+1}{2\gamma+1} \le n < 1$ $y(x, \psi) = Y_1(0) x^n$ $\tag{3.54}$

$$v(x, \psi) = nY_1(0) x^{-(1-n)} \tag{3.55}$$

$\frac{\gamma+2}{3\gamma+2} \le n < \frac{\gamma+1}{2\gamma+1}$ $y(x, \psi) = Y_1(0) x^n + y_2(\psi) x^{-n+\frac{2(1-n)}{\gamma}}$ $\tag{3.56}$

$$v(x, \psi) = nY_1(0) x^{-(1-n)} + \left[\frac{2(1-n)}{\gamma} - n\right] y_2(\psi) x^{-\left(1+n+\frac{2n}{\gamma}-\frac{2}{\gamma}\right)} \tag{3.57}$$

where

$$y_2(\psi) = \frac{k}{Y_1(0)}\left[\frac{2G}{P_1(0)}\right]^{1/\gamma} I(\psi;n,\gamma) + \frac{k\gamma}{Y_1(0)\left(\gamma + 1 - \frac{1}{n}\right)}\left[\frac{G}{P_1(0)}\right]^{1/\gamma} \psi^{\frac{1}{\gamma}\left(\gamma+1-\frac{1}{n}\right)} \tag{3.58}$$

$$I(\psi;n,\gamma) = \int_\psi^\infty \left[\frac{1}{\left(2\psi^{(1-n)/n}\right)^{1/\gamma}} - \frac{1}{\left(1 + 2\psi^{(1-n)/n}\right)^{1/\gamma}}\right] d\psi \tag{3.59}$$

$$\underline{n = \frac{1}{2}} \qquad y(x,\psi) = \left(\frac{\gamma}{\gamma + 1}\right)^{1/2}\left[\frac{2}{(\gamma + 1)P_1(0)}\right]^{\frac{1}{2\gamma}} (2\psi + 1)^{\frac{\gamma-1}{2\gamma}} x^{\frac{1}{2\gamma}} \tag{3.60}$$

$$v(x,\psi) = \frac{1}{2\left[\gamma(\gamma + 1)\right]^{1/2}}\left[\frac{2}{(\gamma + 1)P_1(0)}\right]^{\frac{1}{2\gamma}} (2\psi + 1)^{\frac{\gamma-1}{2\gamma}} x^{-\left(1-\frac{1}{2\gamma}\right)} \tag{3.61}$$

3.3 Composite Expansions

Equation (2.26) can now be applied to the inner and outer solutions resulting in a solution that is uniformly valid from the shock to the body. Thus the final results are:

$$\underline{\frac{1}{2} \leq n < 1} \qquad\qquad p = 2n^2 F(z) x^{-2(1-n)} \tag{3.62}$$

$$\rho = \beta(z) + \left[\frac{P_1(0)}{2G}\right]^{1/\gamma}\left[\left(1 + 2\psi^{\frac{1-n}{n}}\right)^{\frac{1}{\gamma}} - \left(2\psi^{\frac{1-n}{n}}\right)^{\frac{1}{\gamma}}\right] x^{\frac{-2(1-n)}{\gamma}} \tag{3.63}$$

$$u = 1 - n^2\left[\frac{2\gamma}{\gamma - 1} \frac{F(z)}{\beta(z)} + \phi^2(z)\right] x^{-2(1-n)}$$

$$+ \frac{\gamma}{\gamma - 1} P_1(0)\left[\frac{2G}{P_1(0)}\right]^{1/\gamma}\left[\frac{1}{\left(2\psi^{(1-n)/n}\right)^{1/\gamma}} - \frac{1}{\left(1 + 2\psi^{(1-n)/n}\right)^{1/\gamma}}\right] x^{-2(1-n)\frac{\gamma-1}{\gamma}} \tag{3.64}$$

$$\frac{\gamma+1}{2\gamma+1} \leq n < 1 \qquad\qquad y = \sqrt{2}\, zx^n \tag{3.65}$$

$$v = \sqrt{2}\, n\phi(z)\, x^{-(1-n)} \tag{3.66}$$

$$\frac{\gamma+2}{3\gamma+2} \leq n < \frac{\gamma+1}{2\gamma+1} \qquad y = \sqrt{2}\, zx^n + \frac{k}{Y_1(0)}\left[\frac{2G}{P_1(0)}\right]^{1/\gamma} I(\psi;n,\gamma) x^{-n+\frac{2(1-n)}{\gamma}} \tag{3.67}$$

$$v = \sqrt{2}\, n\phi(z)\, x^{-(1-n)} + \left[\frac{2(1-n)}{\gamma} - n\right]\frac{k}{Y_1(0)}\left[\frac{2G}{P_1(0)}\right]^{1/\gamma} I(\psi;n,\gamma)\, x^{\left(1+n+\frac{2n}{\gamma}-\frac{2}{\gamma}\right)} \tag{3.68}$$

$$n = \frac{1}{2} \qquad y = \sqrt{2}\, zx^{1/2} + \left(\frac{\gamma}{\gamma+1}\right)^{1/2}\left[\frac{2}{(\gamma+1)P_1(0)}\right]^{\frac{1}{2\gamma}}\left[(2\psi+1)^{\frac{\gamma-1}{2\gamma}} - (2\psi)^{\frac{\gamma-1}{2\gamma}}\right]x^{\frac{1}{2\gamma}} \tag{3.69}$$

$$v = \frac{\phi(z)}{\sqrt{2}}\, x^{-1/2} + \frac{1}{2\left[\gamma(\gamma+1)\right]^{1/2}}\left[\frac{2}{(\gamma+1)P_1(0)}\right]^{\frac{1}{2\gamma}}\left[(2\psi+1)^{\frac{\gamma-1}{2\gamma}} - (2\psi)^{\frac{\gamma-1}{2\gamma}}\right]x^{-\left(1-\frac{1}{2\gamma}\right)} \tag{3.70}$$

In the above equations the relation between the similarity variable z and the Von Mises variable ψ is given by:

$$\psi = \frac{z\beta(z)}{k}\left[z - \phi(z)\right]x^{2n} \tag{3.71}$$

4. NUMERICAL RESULTS

The body shapes associated with the family of power-law shocks can be obtained by setting $\psi = 0$ in the inner solutions, Equations (3.51) through (3.61). The results are:

$$\frac{\gamma + 1}{2\gamma + 1} \leq n < 1 \qquad\qquad y_b = Y_1(0) x^n \tag{4.1}$$

$$\frac{\gamma + 2}{3\gamma + 2} \leq n < \frac{\gamma + 1}{2\gamma + 1} \qquad y_b = Y_1(0)x^n + \left\{ \frac{k}{Y_1(0)} \left[\frac{2G}{P_1(0)} \right]^{1/\gamma} I(0;n,\gamma) \right\} x^{-n + \frac{2(1-n)}{\gamma}} \tag{4.2}$$

$$n = \frac{1}{2} \qquad\qquad y_b = \left[\left(\frac{\gamma}{\gamma + 1} \right)^{1/2} \left(\frac{2}{(\gamma + 1)P_1(0)} \right)^{\frac{1}{2\gamma}} \right] x^{\frac{1}{2\gamma}} \tag{4.3}$$

The effect of the entropy layer on the body shape can be seen more clearly by defining a parameter as follows:

$$\alpha = \frac{y_b - y_{b_o}}{y_s - y_{b_o}} \tag{4.4}$$

where y_{b_o} is the body shape given by the first order outer solution. In the range of values of n considered the results are:

$$\frac{\gamma + 1}{2\gamma + 1} \leq n < 1 \qquad\qquad \alpha \equiv 0 \tag{4.5}$$

$$\frac{\gamma + 2}{3\gamma + 2} \leq n < \frac{\gamma + 1}{2\gamma + 1} \qquad \alpha = \left\{ \frac{k}{Y_1(0)\left[\sqrt{2} - Y_1(0) \right]} \left[\frac{2G}{P_1(0)} \right]^{1/\gamma} I(0;n,\gamma) \right\} x^{-2n + \frac{2(1-n)}{\gamma}} \tag{4.6}$$

$$n = 1/2 \qquad \alpha = \left[\frac{\gamma}{2(\gamma + 1)} \right]^{1/2} \left[\frac{2}{(\gamma + 1)P_1(0)} \right]^{\frac{1}{2\gamma}} x^{-\frac{\gamma - 1}{2\gamma}} \tag{4.7}$$

Thus, for $n \geq (\gamma + 1)/(2\gamma + 1)$, the entropy layer does not effect the body shape obtained from the first order outer solution. This agrees with

a result previously obtained by Sychev (Reference 3). For $n < (\gamma + 1)/$ $(2\gamma + 1)$, the effect of the entropy layer on the parameter a vanishes for large distances downstream. As $n \to 1/2$, the effect vanishes more slowly in the downstream direction. For example, with $\gamma = 1.4$:

$$n = \frac{2}{3} \qquad\qquad a = 0$$

$$n = \frac{4}{7} \qquad\qquad a \sim x^{-0.520} \qquad\qquad (4.8)$$

$$n = \frac{1}{2} \qquad\qquad a \sim x^{-0.143}$$

The effect of nose bluntness on the inviscid flow field can be illustrated by considering a family of finite length bodies of the same thickness ratio,

$$\tau = \frac{(y_b)_{max}}{\ell} \qquad\qquad (4.9)$$

and different values of the parameter n. A family of such bodies with $\tau = 1/4$ is shown in Figure 2. The flow field profiles at the base station are shown in Figures 3 and 4 plotted versus the proportional distance across the shock layer:

$$y* = \frac{y - y_b}{y_s - y_b} \qquad\qquad (4.10)$$

The existence of the entropy layer is most clearly illustrated by the velocity and entropy function profiles of Figure 4. The more blunt bodies or lower values of n produce thicker entropy layers with smaller gradients at the body surface.
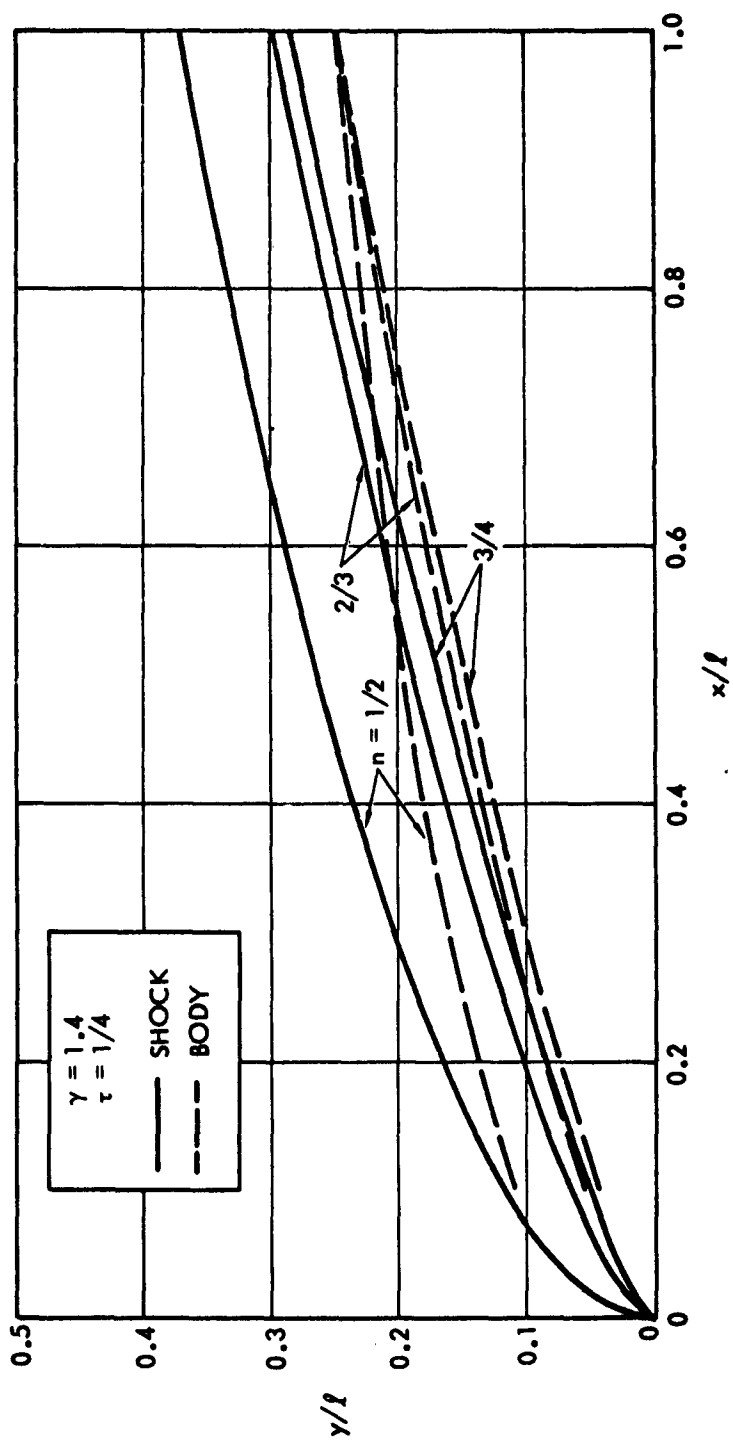
Figure 2. Bodies With Power-Law Shocks
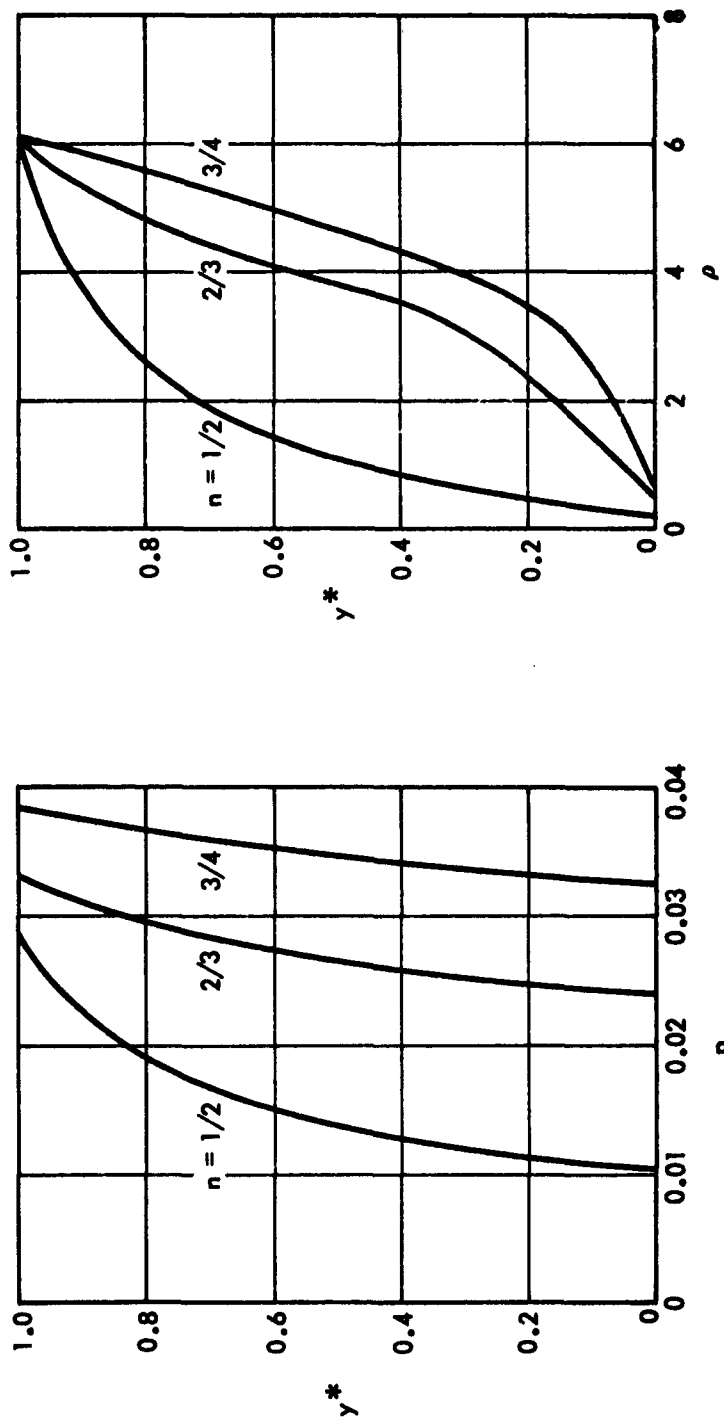
$\gamma = 1.4, \; \tau = 1/4, \; x/\ell = 1$

Figure 3. Inviscid Shock Layer Profiles
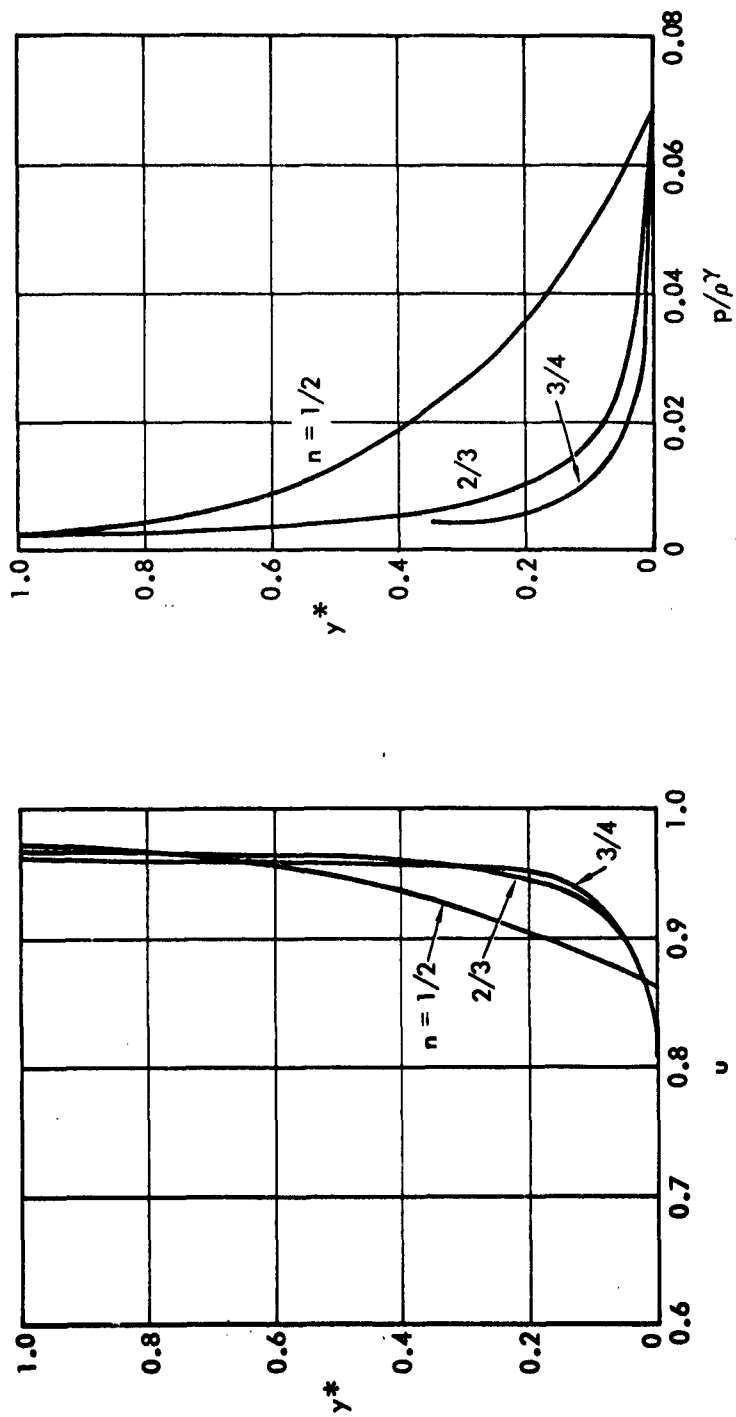
Figure 4. Inviscid Shock Layer Profiles

$\gamma = 1.4$, $\tau = 1/4$, $x/\ell = 1$

5. CONCLUSIONS

The method of inner and outer expansions can be used to treat the entropy layer near the body for blunt nose shpaes with power-law shocks, $y_s \sim x^n$, for $1/2 \leq n < 1$. The outer solutions near the shock wave are identical to the blast wave solutions. The method essentially adds a constant pressure entropy layer to the blast wave solution thus providing a solution that is uniformly valid across the shock layer.

In this report the outer solutions were expanded about the body to obtain the form of the inner expansions for $1/2 \leq n < 1$. However, as pointed out by Lees and Kubota (Reference 1), for $n < 1/2$, the outer solution does not exist. It is not clear how the method of inner and outer expansions can be applied to the case $n < 1/2$ or if such an analytic solution can be obtained.

The inner expansion for the space variable $y(x, \psi)$ consistent with the first order outer solutions is a series which converges very slowly as $n \to 1/2$ from above. The case $n = 1/2$ is singular and requires only one term in the expansion. This case was solved previously by Yakura (Reference 4). Also, for $(\gamma + 1)/(2\gamma + 1) \leq n < 1$, only a single term is required in the expansion and the inner solution is easily obtained. In this range of n, the blast wave profiles are modified by the inner solution but the body shape is unchanged. This minimum value of n above which the blast wave body is correct was previously presented by Sychev (Reference 3). If two terms are retained in the expansion for $y(x, \psi)$ the inner solution can be extended to cover the range $(\gamma + 2)/(3\gamma + 2) \leq n < (\gamma + 1)/(2\gamma + 1)$. However, for n only slightly greater than $1/2$ the method is of limited usefulness due to the slow convergence of the expansion.

Acknowledgements

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